Power towers: solution

This problem can be solved by using modular multiplicative inverses. However, this was not what was expected. The solution presented here does not require extensive knowledge of number theory.

Let us start with a simple observation: for any given $M \geq 2$ and $0 \leq d < M$, the sequence of the results of d^x modulo M, as x increases, will inevitably repeat itself (as there are only M possible results). For example, if we take $M = 10$:

$$\begin{aligned}
3^1 &= 3 \equiv 3 \pmod{10}\\
3^2 &= 9 \equiv 9 \pmod{10}\\
3^3 &= 27 \equiv 7 \pmod{10}\\
3^4 &= 81 \equiv 1 \pmod{10}\\
3^5 &= 243 \equiv 3 \pmod{10}\\
&\dots
\end{aligned}$$

and therefore for $d = 3$ and $M = 10$ the sequence is $3, 9, 7, 1, 3, 9, 7, 1, \dots$

In some cases, the sequence is even simpler:

$$\begin{aligned}
6^1 &= 6 \equiv 6 \pmod{10}\\
6^2 &= 36 \equiv 6 \pmod{10}\\
6^3 &= 216 \equiv 6 \pmod{10}\\
&\dots
\end{aligned}$$

and therefore for $d = 6$ and $M = 10$ the sequence is $6, 6, 6, 6, \dots$

In other cases, the first few terms are not part of the sequence's period:

$$\begin{aligned}
3^1 &= 3 \equiv 3 \pmod{36}\\
3^2 &= 9 \equiv 9 \pmod{36}\\
3^3 &= 27 \equiv 27 \pmod{36}\\
3^4 &= 81 \equiv 9 \pmod{36}\\
3^5 &= 243 \equiv 27 \pmod{36}\\
&\dots
\end{aligned}$$

and therefore for $d = 3$ and $M = 36$ the sequence is $3, 9, 27, 9, 27, \dots$

In general, for any given $M \geq 2$ and $0 \leq d < M$, we can calculate this sequence of remainders of the powers of d modulo M. This takes $O(M)$ time and space and we can find the length k of the sequence's prefix, which is not repeated, and the length n of the sequence's period. For example, for $d = 6$ and $M = 36$, we have $k = 1$ (the term 3 in the beginning of the sequence) and $n = 2$ (the terms $9, 27$ that repeat themselves ad infinitum). The numbers k and n are obviously smaller than M.

Now, let us calculate the power tower:

$$x_1^{x_2^{\cdot^{\cdot^{\cdot^{x_n}}}}} \mod M$$

We first take $d = x_1 \mod M$ and we notice that the result will be the same if we replace x_1 with d. Then, we calculate the sequence S of remainders of the powers of d modulo M, as discussed before, and let k and n be the lengths of its non-repeating prefix and its period, respectively. To arrive at the solution, we just need to calculate recursively the power tower:

$$x_2^{\cdot^{\cdot^{\cdot^{x_n}}}} \mod n$$

Its result will help us find the term of the sequence S that is the answer to our problem. (In practice, it is a bit more complicated, because we have to take into account the case when the result of the smaller power tower does not exceed k.)

This idea alone is enough to solve subtasks 1 to 3. In fact, subtask 1 can be solved by just computing the result using 64-bit integers, and subtask 2 with $M = 10$ is simpler than the general case, as the sequences can easily be precomputed by hand, and they all have $k = 0$ and $n \in \{1, 2, 4\}$.

For full credit, (at least one of) two optimizations were required:

1. Pre-compute parts of the power tower whose result was representable by a 64-bit integer. This makes it much simpler to test if the result of the smaller power tower does not exceed k. It also provides shortcuts in the case that the power tower contains 1's.
2. Use memoization to avoid re-computing the same sequences of remainders.